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SUFFICIENT CONDITIONS FOR MEROMORPHIC STARLIKENESS AND CLOSE-TO-CONVEXITY OF ORDER α

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ABSTRACT. The object of the present paper is to drive a property of certain meromorphic functions in the punctured unit disk. Our main theorem contains certain sufficient conditions for starlikeness and close-to-convexity of order α of meromorphic functions.

1. Introduction

Let Σ denote the class of functions of the form

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n$$

which are analytic in the punctured unit disk $\mathcal{D}=\{z:0<|z|<1\}$. A function $f\in\Sigma$ is said to be meromorphic starlike of order α if it satisfies

$$-\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha\quad(z\in\mathcal{U}=\mathcal{D}-\{0\})$$

for some $\alpha(0\leq\alpha<1)$. We denote $\Sigma^*(\alpha)$ the class of all meromorphic starlike functions of order α.

Let $MC(\alpha)$ be the subclass of Σ consisting of functions f which satisfy

$$-\mathrm{Re}\{z^2f'(z)\}>\alpha\quad(z\in\mathcal{U})$$

for some $\alpha(0\leq\alpha<1)$. A function f in $MC(\alpha)$ is meromorphic close-to-convex of order α in $\mathcal{D}$ [1].

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2. Main result

In proving our main theorem, we need the following lemma due to Owa, Nunokawa, Saitoh and Fukui [2].

Lemma 2.1. *Let p be analytic in $\mathcal{U}$ with $p(0) = 1$. Suppose that there exists a point $z_0 \in \mathcal{U}$ such that $Rep(z) > 0(|z| < |z_0|)$, $Rep(z_0) = 0$, and $p(z) \neq 0$. Then we have $p(z) = ia(a \neq 0)$ and*

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\frac{k}{2}(a + \frac{1}{a}),$$

k is a real number with $k \geq 1$.

With the aid of above Lemma 2.1, we drive

Theorem 2.1. *If $f \in \Sigma$ satisfies $f(z)f'(z) \neq 0$ in $\mathcal{D}$ and*

$$\mathrm{Re}\left\{\alpha\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}\right\} < 2(2-\alpha) - \beta \quad (z \in \mathcal{U}),$$

then

$$-\mathrm{Re}\left\{\frac{z^{2-\alpha}f'(z)}{f^{\alpha}(z)}\right\} > \frac{1}{1 + 2(2-\alpha) - 2\beta} \quad (z \in \mathcal{U}),$$

where $\alpha \leq 2$ and $\frac{2(2-\alpha)-1}{2} \leq \beta < 2 - \alpha$.

Proof. We define the function p in U by

$$-\frac{z^{2-\alpha}f'(z)}{f^{\alpha}(z)} = \gamma + (1-\gamma)p(z)$$

with $\gamma = \frac{1}{1+2(2-\alpha)-2\beta}$. Then p is analytic in $\mathcal{U}$ with $p(0) = 1$ and

$$\alpha\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} = 2 - \alpha - \frac{(1-\gamma)zp'(z)}{\gamma + (1-\gamma)p(z)}.$$

Suppose that there exists a point $z_0 \in \mathcal{U}$ such that

$$\mathrm{Re}p(z) > 0(|z| < |z_0|), \mathrm{Re}p(z_0) = 0, \quad \text{and} \quad p(z) \neq 0.$$

Then, applying Lemma 2.1, we have $p(z) = ia(a \neq 0)$ and

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\frac{k}{2}(a + \frac{1}{a}) \quad (k \geq 1).$$

It follows from this that

$$\begin{aligned}\alpha\frac{z_0f'(z_0)}{f(z_0)}-\frac{z_0f''(z_0)}{f'(z_0)}&=2-\alpha-\frac{(1-\gamma)z_0p'(z_0)}{\gamma+(1-\gamma)p(z_0)}\\&=2-\alpha+\frac{k(1-\gamma)(1+a^2)}{2(\gamma+i(1-\gamma)a)}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\mathrm{Re}\left\{\alpha\frac{z_0f'(z_0)}{f(z_0)}-\frac{z_0f''(z_0)}{f'(z_0)}\right\}&=2-\alpha+\frac{k(1-\gamma)(1+a^2)}{2(\gamma^2+(1-\gamma)^2a^2)}\\&\geq 2-\alpha+\frac{k(1-\gamma)}{2\gamma}\\&\geq 2(2-\alpha)-\beta.\end{aligned}$$

This contracts our assumption. Thus, we conclude that $\mathrm{Re}p(z)>0$ for all $z\in\mathcal{U}$, that is, that

$$-\mathrm{Re}\left\{\frac{z^{2-\alpha}f'(z)}{f^{\alpha}(z)}\right\}>\gamma=\frac{1}{1+2(2-\alpha)-2\beta}\quad(z\in U).$$

Putting $\beta=\frac{2(2-\alpha)-1}{2}$ in Theorem 2.1, we have

Corollary 2.1. *If $f\in\Sigma$ satisfies $f(z)f'(z)\neq 0$ in$\mathcal{D}$ and*

$$\mathrm{Re}\left\{\alpha\frac{zf'(z)}{f(z)}-\frac{zf''(z)}{f'(z)}\right\}<\frac{3}{2}-\alpha\quad(z\in\mathcal{U}),$$

then

$$-\mathrm{Re}\left\{\frac{z^{2-\alpha}f'(z)}{f^{\alpha}(z)}\right\}>\frac{1}{2}\quad(z\in\mathcal{U}),$$

where $\alpha\leq 2$.

Taking $\alpha=1$ in Theorem 2.1, we have

Corollary 2.2. *If $f\in\Sigma$ satisfies $f(z)f'(z)\neq 0$ in $\mathcal{D}$ and*

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}-\frac{zf''(z)}{f'(z)}\right\}<2-\beta\quad(z\in\mathcal{U}),$$

then

$$-\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1}{3-2\beta} \quad (z \in \mathcal{U}),$$

that is, $f \in \Sigma^*\left(\frac{1}{3-2\beta}\right)$, *where* $\frac{1}{2} \leq \beta < 1$.

Further, letting $\alpha = 0$ in Theorem, we have

Corollary 2.3. *If* $f \in \Sigma$ *satisfies* $f(z)f'(z) \neq 0$ *in* $\mathcal{D}$ *and*

$$-\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} < 4-\beta \quad (z \in \mathcal{U}),$$

then

$$-\mathrm{Re}\left\{z^2 f'(z)\right\} > 5-2\beta \quad (z \in \mathcal{U}),$$

that is, $f \in MC\left(\frac{1}{5-2\beta}\right)$, *where* $\frac{3}{2} \leq \beta < 2$.

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References

1. M.D. Ganigi and B.A. Uraligaddi, *Subclasses of meromorphic close-to-convex functions* , Bull. Math. Soc. Sci. Math. R.S. Roumanie(N.S.)**33(81)**(1989), 105-109.
2. S. Owa, M. Nunokawa, H. Saitoh and S. FuKui, *Starlikeness and close-to-convexity of certain analytic functions* , Far East J. Math. Sci. **2**(2)(1994), 143-148.

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